Journal of Statistical Physics, Vol. 71, Nos. 3/4, 1993

Probability Distributions for the Overlaps and Self-Correlations of the Pure States of an n-Vector Model

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Received November 2, 1992

The probability distributions for the overlaps between and the self-correlations of the pure states of the Stanley n-vector model with infinite-range interactions are derived. These probability distributions represent two new order parameters for the model and are intimately related to the parameters which arise naturally within the replica formalism for the treatment of the corresponding quenched random-bond model. In contrast to the $n=1$ Ising case, the probability distributions are nontrivial when $n>1$ and an additional parameter for self-correlation has to be introduced.

KEY WORDS: n-vector model; pure states; correlation functions; random spin systems.

1. INTRODUCTION

In 1968, H. E. Stanley[1] introduced the n-vector model as a unifying description of many simpler nonrandom models in statistical mechanics such as the Ising model ($n=1$), the Vaks–Larkin plane rotator model ($n=2$), the classical Heisenberg model ($n=3$), and the Berlin–Kac spherical model ($n=\infty$). Stanley's exact solutions are confined to nearest neighbor one-dimensional chains and hence do not exhibit a phase transition.

On the other hand, the mean field theory obtained by considering this model with an infinite-range potential introduces a phase transition. The infinite-range model is described by the Hamiltonian

$$-\beta\mathscr{H}=\frac{2J}{N}\sum_{1\leqslant i<j\leqslant N}\mathbf{S}_i^T\mathbf{S}_j+\mathbf{B}^T\sum_{i=1}^{N}\mathbf{S}_i \tag{1.1}$$

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0022-4715/93/0500-0361$07.00/0

where the $\mathbf{S}_i^T \equiv (S_i^\alpha) \equiv (S_i^1,..., S_i^n)$ are classical n-vectors normalized to $\|\mathbf{S}_i\| = \sqrt{n}$, α, β denote running indices for the vector components, i, j denote lattice sites, $\mathbf{V}^T$ is the transposed vector $\mathbf{V}$, and $\mathbf{B}$ is an external magnetic field. The main thermodynamic features of this model have been derived by Silver *et al.*[2] by means of a random walk probability distribution.

If one generalizes the n-vector model (1.1) by introducing quenched random bonds, then it turns out that the only adequate description of the model is provided by two order parameters. The first one is the probability distribution $P_{\alpha\beta}$ for the overlap $q_{\alpha\beta}$ between two of its pure states, while the second one is the probability distribution $W_{\alpha\beta}$ for the self-correlation $d_{\alpha\beta}$ of a pure state of the system.[3] The quantities $q_{\alpha\beta}$, $d_{\alpha\beta}$ and $P_{\alpha\beta}$, $W_{\alpha\beta}$ are defined in Eqs. (3.1)–(3.2).

It is therefore interesting to see what $P_{\alpha\beta}$ and $W_{\alpha\beta}$ look like for the exactly solvable nonrandom model. Furthermore, $P_{\alpha\beta}$ and $W_{\alpha\beta}$ also provide an unusual description of the n-vector model (1.1) itself. Instead of describing the thermodynamic phases (pure states) of the model by means of an external magnetic field $\mathbf{B}$, its space of pure states is characterized by the probability distributions $P_{\alpha\beta}$ and $W_{\alpha\beta}$.

As a final spinoff, this formulation requires the introduction of a new and interesting order parameter, the correlation function for the vector components at one site, $\langle S_i^\alpha S_i^\beta \rangle$. We shall call this $n \times n$ matrix $\langle S_i^\alpha S_i^\beta \rangle$ the *self-correlation* of $\mathbf{S}$. It turns out that $\langle S_i^\alpha S_i^\beta \rangle$ can be expressed as a simple function of J without having to solve any transcendental equations as for the magnetization. Yet $\langle S_i^\alpha S_i^\beta \rangle$ reveals all the characteric thermodynamic behavior of the system at a glimpse, as can be seen from Eq. (4.14).

The paper is organized as follows. In Section 2, which is pedagogical, we briefly review the concept of a pure state with which most readers are probably familiar but which might help some readers in the perusal of the subsequent sections.

In Section 3 we then define the overlap $q_{\alpha\beta}$ between two pure states and the self-correlation $d_{\alpha\beta}$ for a pure state of our model (1.1). The overlap $q_{\alpha\beta}$ is determined by the average magnetization per site $\langle S_i^\alpha \rangle$ and is simply a generalization of the definition for overlap used within the Parisi theory of spin glasses. The self-correlation $d_{\alpha\beta}$ of a pure state, on the other hand, is a new order parameter which appears only for $n > 1$. It depends on the self-correlation of $\mathbf{S}$, $\langle S_i^\alpha S_i^\beta \rangle$. We also present alternative definitions q and d for the overlap and self-correlation of the pure states of the nonrandom n-vector model which are intuitive but which average out some of the information contained in $q_{\alpha\beta}$ and $d_{\alpha\beta}$. Finally, we define the probability distributions $P_{\alpha\beta}$, $W_{\alpha\beta}$, P, and W.

In Section 4 we briefly state the Silver *et al.*[2] result for the magnetization per site $\langle S_i^\alpha \rangle$ and then proceed to derive the self-correlation $\langle S_i^\alpha S_i^\beta \rangle$ for our model.

The remaining Sections 5–7 are devoted to the evaluation of the probability distributions $P_{\alpha\beta}$, $W_{\alpha\beta}$, and P. In all cases, we have managed to express them analytically and for general n in terms of hypergeometric functions ${}_2F_1$, or, alternatively, associated Legendre functions of the second kind, Q_ν.

2. PURE STATES

In ordinary equilibrium statistical mechanics, statistical expectation values for an observable $\mathcal{O}$ are calculated as

$$\langle \mathcal{O}(S) \rangle \equiv \frac{\sum_{\{S\}} \mathcal{O}(S) \exp[-\beta \mathcal{H}(S)]}{\sum_{\{S\}} \exp[-\beta \mathcal{H}(S)]} \tag{2.1}$$

where $\mathcal{H}$ is the Hamiltonian of the system and $\{S\}$ some set of statistical variables.

If we take the example of an Ising spin system without magnetic field, and take $\mathcal{O}$ to be the magnetization per site, then we realize that the prescription (2.1) conceals a lot of the physics for low temperatures. It predicts zero magnetization, whereas we know that at low temperatures the system experiences a spontaneous symmetry breaking which leads to spontaneous magnetization. Equation (2.1) does not describe a pure state (thermodynamic phase) of the system, but a mixture of different states, the Gibbs state. We can decompose the Gibbs state as the sum of K pure equilibrium states which occur with probability P_r,

$$\langle \mathcal{O}(S) \rangle = \sum_{r=1}^{K} P_r \langle \mathcal{O}(S) \rangle_r, \qquad \sum_{r=1}^{K} P_r = 1 \tag{2.2}$$

where $\langle \cdot \rangle_r$ denotes the expectation value for the pure state r. In the case of the Ising system without magnetic field we have two pure states, one with positive magnetization, the other with negative magnetization, and $P_1 = P_2 = 1/2$. We interpret Eq. (2.2) as the system in equilibrium occupying one of the pure states (depending on its history), and the time needed to go from one equilibrium state to another growing exponentially with the size of the system.

The difference between the Gibbs state and pure states can be characterized by means of the connected correlation function

$$\langle S_i S_j \rangle - \langle S_i \rangle \langle S_j \rangle \tag{2.3}$$

For pure states, the connected correlation function must vanish at large distances, whereas for the Gibbs state it does not. This means that as $N \to \infty$ intensive quantities like $(1/N) \sum_{i=1}^{N} S_i$ do not fluctuate in a pure state, but they do fluctuate within a mixture of pure states (clustering). If we consider the example of the Curie–Weiss model, we obtain two pure states characterized by

$$\langle S_i \rangle = \pm m, \qquad \langle S_i S_j \rangle = m^2, \qquad \lim_{(i-j) \to \infty} (\langle S_i S_j \rangle - \langle S_i \rangle \langle S_j \rangle) = 0 \tag{2.4}$$

while for the Gibbs state we have

$$\langle S_i \rangle = 0, \quad \langle S_i S_j \rangle = m^2, \quad \lim_{(i-j) \to \infty} (\langle S_i S_j \rangle - \langle S_i \rangle \langle S_j \rangle) = m^2 \neq 0 \tag{2.5}$$

In the case of nonrandom Ising spin models, we can extract the two pure states by simply introducing a uniform, external magnetic field $\mathbf{B}$ which may then go to zero as $N \to \infty$. Depending on the sign of $\mathbf{B}$, this prepares the system either in the pure state with $\langle S_i \rangle = m$ or in the pure state with $\langle S_i \rangle = -m$.

In the case of random spin systems, which this approach is unfortunately not possible since the pure states are not related by any apparent symmetry. A magnetic field which prepares a pure state would have to be site dependent and follow the local spontaneous magnetizations. We have to know these local spontaneous magnetizations before we can define such a field. However, the pure states can still be defined without using any auxiliary fields by means of the connected correlation function (2.3). More details on the concept of pure states can be found, e.g., in refs. 4.

3. DEFINITION OF THE OVERLAP AND SELF-CORRELATION PARAMETERS

We now return to the n-vector model (1.1). Below T_c, and in zero magnetic field, the system will occupy one out of a continuum of pure states if $n > 1$. Each of these pure states can be characterized by a unit vector $\hat{\mathbf{B}}$ and the limiting procedure $\mathbf{B} \to 0$. The continuum of pure states maps onto the n-dimensional unit sphere. We can describe the space of pure states without recourse to an external field by defining the overlaps $q_{\alpha\beta}(r, t)$ [or $q(r, t)$] between two pure states r and t and the self-correlations $d_{\alpha\beta}(r)$ [or $d(r)$] for a single pure state,

$$
\begin{aligned}
q_{\alpha\beta}(r, t) &\equiv \frac{1}{N} \sum_{i=1}^{N} \langle S_i^\alpha \rangle_r \langle S_i^\beta \rangle_t \\
d_{\alpha\beta}(r) &\equiv \frac{1}{N} \sum_{i=1}^{N} \langle S_i^\alpha S_i^\beta \rangle_r \\
q(r, t) &\equiv \frac{1}{N} \sum_{i=1}^{N} \langle \mathbf{S}_i \rangle_i^T \langle \mathbf{S}_i \rangle_t \\
d(r) &\equiv \frac{1}{N} \sum_{i=1}^{N} \langle \mathbf{S}_i^T \mathbf{S}_i \rangle_r = n
\end{aligned}
\tag{3.1}
$$

$\langle \cdot \rangle_r$ refers to the thermal average restricted to the pure state r.

The definitions for overlap $q_{\alpha\beta}$ and for self-correlation $d_{\alpha\beta}$ corresponds to parameters which arise naturally within the replica formalism for the quenched random-bond n-vector model.[3] The definitions for q and d, on the other hand, though intuitive, average out some of the information contained in $q_{\alpha\beta}$ and $d_{\alpha\beta}$. Both sets of definitions represent generalizations of the definition for overlap which is used in the Parisi theory of spin glasses and to which they reduce for $n = 1$. In particular, the parameter for self-correlation $d_{\alpha\beta}$ which is necessary in the replica theory for $n > 1$ becomes equal to one when $n = 1$ and hence does not appear as a parameter in the Parisi theory at all. We shall investigate both sets of definitions for the nonrandom model.

If we know the probability P_r for the system to settle into the pure state r, then we can describe the structure of the space of pure states by giving the probability distributions for the above overlaps and self-correlations,

$$
\begin{aligned}
P_{\alpha\beta}(q_{\alpha\beta}) &= \sum_{r,t} P_r P_t \delta[q_{\alpha\beta} - q_{\alpha\beta}(r, t)] \\
W_{\alpha\beta}(d_{\alpha\beta}) &= \sum_{r} P_r \delta[d_{\alpha\beta} - d_{\alpha\beta}(r)] \\
P(q) &= \sum_{r,t} P_r P_t \delta[q - q(r, t)] \\
W(d) &= \sum_{r} P_r \delta[d - d(r)] = \delta(d - n)
\end{aligned}
\tag{3.2}
$$

As we mentioned in the previous section, since the pure states of the quenched n-vector model are not related by any apparent symmetry, they cannot be extracted by some external magnetic field, and the distributions (3.2) are the only way of describing its space of pure states.

In the case of the nonrandom n-vector model, the probability distributions (3.2) can be evaluated exactly. The solutions can be expressed in terms of hypergeometric functions ${}_2F_1$, or alternatively, in terms of associated Legendre functions of the second kind, Q_ν. Furthermore, as we mentioned in the introduction, they describe precisely the geometrical degeneracy of the overlap and self-correlation parameters which appear in the replica formalism for the quenched n-vector model.[3]

4. MAGNETIZATION PER SITE $\langle S_i^\alpha \rangle$ AND SELF-CORRELATION $\langle S_i^\alpha S_i^\beta \rangle$

The magnetization per site has been evaluated by Silver *et al.*[3] They find

$$\langle S_i^\alpha \rangle = \sqrt{n}\, x_s \hat{B}^\alpha \equiv m\hat{B}^\alpha \tag{4.1}$$

where $\hat{B}^\alpha$ is the α component of the unit vector $\hat{\mathbf{B}}$ in the direction of $\mathbf{B}$, and where x_s is a solution of the transcendental order parameter equation

$$x_s = \frac{I_{n/2}(2Jnx_s + B\sqrt{n})}{I_{n/2-1}(2Jnx_s + B\sqrt{n})} \tag{4.2}$$

The I_ν are modified Bessel functions of order ν. There is a critical temperature T_c corresponding to the critical point $J_c = 1/2$. Solutions $m > 0$ exist only for $T > T_c$.

In order to evaluate the self-correlation $\langle S_i^\alpha S_i^\beta \rangle$, on the other hand, we first apply the relation

$$e^{\mathbf{a}^T\mathbf{a}/2} = \int_{-\infty}^{\infty} D\mathbf{y}\, e^{\mathbf{y}^T\mathbf{a}}, \qquad D\mathbf{y} \equiv \frac{d\mathbf{y}}{(2\pi)^{n/2}}\, e^{-\mathbf{y}^T\mathbf{y}/2} \tag{4.3}$$

to the definition of $\langle S_i^\alpha S_i^\beta \rangle$. With the partition function Z_N and the abbreviation

$$\boldsymbol{\eta} \equiv \left(\frac{2J}{N}\right)^{1/2} \mathbf{y} + \mathbf{B} \tag{4.4}$$

we have

$$\langle S_i^\alpha S_i^\beta \rangle \equiv \mathrm{Tr}_{\{\mathbf{S}_k\}} \left[S_i^\alpha S_i^\beta \exp\left(\frac{J}{N} \sum_{j,k=1}^{N} \mathbf{S}_j^T \mathbf{S}_k + \mathbf{B}^T \sum_{j=1}^{N} \mathbf{S}_j - Jn \right) \right] \Big/ Z_N$$

$$= \frac{\begin{bmatrix} \int_{-\infty}^{\infty} D\mathbf{y} \int_{\|\mathbf{S}\| = \sqrt{n}} S^\alpha S^\beta \exp(\boldsymbol{\eta}^T\mathbf{S})\, d\mathbf{S} \\ \cdot \left[\int_{\|\mathbf{S}\| = \sqrt{n}} \exp(\boldsymbol{\eta}^T\mathbf{S})\, d\mathbf{S} \right]^{N-1} \end{bmatrix}}{\int_{-\infty}^{\infty} D\mathbf{y} \int_{\|\mathbf{S}\| = \sqrt{n}} \exp(\boldsymbol{\eta}^T\mathbf{S})\, d\mathbf{S} \cdot \left[\int_{\|\mathbf{S}\| = \sqrt{n}} \exp(\boldsymbol{\eta}^T\mathbf{S})\, d\mathbf{S} \right]^{N-1}} \tag{4.5}$$

The integrals in this expression can be evaluated using the relations

$$\int_{\|\mathbf{S}\|=R} S^\alpha S^\beta \exp(\boldsymbol{\eta}^T \mathbf{S})\, d\mathbf{S}$$
$$= R^{n+1}(2\pi)^{n/2}(\eta R)^{-(n-2)/2}\left[\frac{\delta_{\alpha\beta}}{\eta R} I_{n/2}(\eta R) + \frac{\eta^\alpha \eta^\beta}{\eta^2} I_{n/2+1}(\eta R)\right] \tag{4.6}$$
$$\int_{\|\mathbf{S}\|=R} \exp(\boldsymbol{\eta}^T \mathbf{S})\, d\mathbf{S}$$
$$= R^{n-1}(2\pi)^{n/2}(\eta R)^{-(n-2)/2} I_{n/2-1}(\eta R)$$

where $\delta_{\alpha\beta}$ is the Kronecker delta symbol and where $\eta \equiv \|\boldsymbol{\eta}\|$. We find

$$\langle S_i^\alpha S_i^\beta \rangle = n \int_{-\infty}^{\infty} D\mathbf{y} \left[\delta_{\alpha\beta} \frac{I_{n/2}(\sqrt{n}\,\eta)}{(\sqrt{n}\,\eta)^{n/2}} + \frac{\eta^\alpha \eta^\beta}{\eta^2} \frac{I_{n/2+1}(\sqrt{n}\,\eta)}{(\sqrt{n}\,\eta)^{n/2-1}}\right]$$
$$\times \left[\frac{I_{n/2-1}(\sqrt{n}\,\eta)}{(\sqrt{n}\,\eta)^{n/2-1}}\right]^{N-1}$$
$$\times \left\{\int_{-\infty}^{\infty} D\mathbf{y} \frac{I_{n/2-1}(\sqrt{n}\,\eta)}{(\sqrt{n}\,\eta)^{n/2-1}} \left[\frac{I_{n/2-1}(\sqrt{n}\,\eta)}{(\sqrt{n}\,\eta)^{n/2-1}}\right]^{N-1}\right\}^{-1}$$
$$= n \int_{-\infty}^{\infty} D\mathbf{x} \left[\delta_{\alpha\beta} \frac{I_{n/2}(\xi)}{(\xi)^{n/2}} + \frac{\xi^\alpha \xi^\beta}{\xi^2} \frac{I_{n/2+1}(\xi)}{(\xi)^{n/2-1}}\right] \exp[NG(\mathbf{x}, \mathbf{B})]$$
$$\times \left\{\int_{-\infty}^{\infty} D\mathbf{x} \frac{I_{n/2-1}(\xi)}{(\xi)^{n/2-1}} \exp[NG(\mathbf{x}, \mathbf{B})]\right\}^{-1} \tag{4.7}$$

where in the last step we have performed the coordinate transformation $\mathbf{y} \equiv (2JnN)^{1/2}\,\mathbf{x}$ and we have defined

$$\boldsymbol{\xi} = 2Jn\mathbf{x} + \mathbf{B}\sqrt{n}, \qquad \xi \equiv \|\boldsymbol{\xi}\| \tag{4.8}$$

and

$$G(\mathbf{x}, \mathbf{B}) \equiv -Jn\mathbf{x}^T\mathbf{x} + \ln \frac{I_{n/2-1}(\|2Jn\mathbf{x} + \mathbf{B}\sqrt{n}\|)}{\|2Jn\mathbf{x} + \mathbf{B}\sqrt{n}\|^{n/2-1}} \tag{4.9}$$

As can be seen from Eq. (9.6.10) in ref. 5, the function $I_{n/2-1}(z)/z^{n/2-1}$ increases monotonically for real $z \geqslant 0$. Therefore, the right-hand side of Eq. (4.9) will assume its maximum value when $\mathbf{x}$ is parallel to $\mathbf{B}$. The saddle point method of Laplace applied to the integrals in Eq. (4.7) as $N \to \infty$ then yields

$$\langle S_i^\alpha S_i^\beta \rangle = n \left[\frac{\delta_{\alpha\beta}}{(2Jnx_s + B\sqrt{n})} \frac{I_{n/2}(2Jnx_s + B\sqrt{n})}{I_{n/2-1}(2Jnx_s + B\sqrt{n})} + \hat{B}^\alpha \hat{B}^\beta \frac{I_{n/2+1}(2Jnx_s + B\sqrt{n})}{I_{n/2-1}(2Jnx_s + B\sqrt{n})} \right] \tag{4.10}$$

where x_s represents the saddle point which maximizes

$$G(x, b) = -Jnx^2 + \ln \frac{I_{n/2-1}(2Jnx_s + B\sqrt{n})}{(2Jnx_s + B\sqrt{n})^{n/2-1}} \tag{4.11}$$

The saddle point equation $\partial G/\partial x_s = 0$ reproduces exactly the order parameter equation (4.2) and thus justifies *post facto* our notation x_s.

By using the relation

$$\frac{n}{2} I_{n/2}(z) + I_{n/2+1}(z) = I_{n/2-1}(z) \tag{4.12}$$

and Eq. (4.2), we can write the self-correlation (4.10) as

$$\langle S_i^\alpha S_i^\beta \rangle = \frac{nx_s}{2Jnx_s + B\sqrt{n}} \delta_{\alpha\beta} + n\hat{B}^\alpha \hat{B}^\beta \left(1 - \frac{nx_s}{2Jnx_s + B\sqrt{n}}\right) \tag{4.13}$$

In the limit $B \to 0$, we have $x_s \approx B/[\sqrt{n}(1-2J)]$ when $T > T_c$ and $x_s \neq 0$ when $T < T_c$. Therefore, we find

$$\langle S_i^\alpha S_i^\beta \rangle \xrightarrow{B \to 0} \begin{cases} \delta_{\alpha\beta} & (T > T_c) \\ \frac{1}{2J} \delta_{\alpha\beta} + n\hat{B}^\alpha \hat{B}^\beta \left(1 - \frac{1}{2J}\right) & (T < T_c) \end{cases} \tag{4.14}$$

This equation is understood as the magnitude B of $\mathbf{B}$ approaching zero while the direction $\hat{\mathbf{B}}$ remains frozen in.

We see that $\langle S_i^\alpha S_i^\beta \rangle$ can be expressed explicitly as a function of J without having to solve a transcendental equation. For $T > T_c$, the spins decouple and the self-correlation is $\delta_{\alpha\beta}$. As $T \to 0$, the spins are completely correlated with the external magnetic field and the self-correlation becomes $n\hat{B}^\alpha \hat{B}^\beta$. Finally, the phase transition point $J_c = 1/2$ can be read off immediately from (4.14).

We conclude this section by noting that as $B \to 0$ the model becomes isotropic and the α dependence in (4.1) disappears, i.e., $q_{\alpha\beta}$ and hence $P_{\alpha\beta}$ must be independent of $\alpha\beta$. In the same fashion, $\langle S_i^\alpha S_i^\beta \rangle$ and hence $d_{\alpha\beta}$, $W_{\alpha\beta}$ will not depend on α and β for $\alpha \neq \beta$ and not on α for $\alpha = \beta$.

5. PROBABILITY DISTRIBUTION $P_{\alpha\beta}$ FOR THE OVERLAP $q_{\alpha\beta}$

From Eq. (4.1), we find that the overlap $q_{\alpha\beta}(r, t)$ defined in (3.1) is given by

$$q_{\alpha\beta}(r, t) = \frac{1}{N} \sum_{i=1}^{N} \langle S_i^\alpha \rangle_r \langle S_i^\beta \rangle_t = \begin{cases} 0 & (T > T_c) \\ m^2 \hat{B}_r^\alpha \hat{B}_t^\beta & (T < T_c) \end{cases} \tag{5.1}$$

$m = \sqrt{n}\, x_s$ refers to the solution of (4.2) as $B \to 0$. Here $\hat{\mathbf{B}}_r$ and $\hat{\mathbf{B}}_t$ are simply two unit vectors, parametrized by the symbolic subscripts r and t. We explained at the beginning of Section 3 that each pure state r of our model can be represented by an n-dimensional unit vector $\hat{\mathbf{B}}_r$.

Because of the symmetry of our model when $B \to 0$, each pure state is equally likely. The probability P_r of finding a pure state in the solid angle element $d\Omega$ is therefore given by

$$P_r = \frac{d\Omega}{[2(\pi^{n/2}/\Gamma(n/2))]} \tag{5.2}$$

5.1. $P_{\alpha\beta}(q_{\alpha\beta})$ for $T > T_c$

By inserting (5.1) and (5.2) into the definition (3.2) for $P_{\alpha\beta}$ we get

$$P_{\alpha\beta}(q_{\alpha\beta}) = \sum_{r,t} P_r P_t \delta(q_{\alpha\beta} - 0) = \delta(q_{\alpha\beta}) \tag{5.3}$$

5.2. $P_{\alpha\beta}(q_{\alpha\beta})$ for $T < T_c$

By inserting (5.1) and (5.2) into the definition (3.2) for $P_{\alpha\beta}$, and by using n-dimensional spherical coordinates, we find

$$\begin{aligned} P_{\alpha\beta}(q_{\alpha\beta}) &= \sum_{r,t} P_r P_t \delta(q_{\alpha\beta} - m^2 \hat{B}_r^\alpha \hat{B}_t^\beta) \\ &= \left[\frac{\Gamma(n/2)}{2\pi^{n/2}}\right]^2 \int_0^{2\pi} d\theta_1 \int_0^{2\pi} d\psi_1 \\ &\quad \times \prod_{k=2}^{n-1} \int_0^{\pi} d\theta_k \sin^{k-1}\theta_k \int_0^{\pi} d\psi_k \sin^{k-1}\psi_k \\ &\quad \times \delta(q_{\alpha\beta} - m^2 \cos\theta_{n-1} \cos\psi_{n-1}) \end{aligned} \tag{5.4}$$

where, without loss of generality, we have chosen the x_n axis of the θ coordinate system in the direction α and the x_n axis of the ψ coordinate system in the direction β. After performing the integrations up to θ_{n-2} and ψ_{n-2}, we have

$$P_{\alpha\beta}(q_{\alpha\beta}) = \frac{\Gamma(n/2)^2}{\pi\Gamma((n-1)/2)^2} \int_0^\pi d\theta \int_0^\pi d\psi \sin^{n-2}\theta \sin^{n-2}\psi$$
$$\times\ \delta(q_{\alpha\beta} - m^2 \cos\theta \cos\psi)$$
$$= \frac{\Gamma(n/2)^2}{\pi\Gamma((n-1)/2)^2} \int_{-1}^1 dx \int_{-1}^1 dy\, (1-x^2)^{(n-3)/2}$$
$$\times\ (1-y^2)^{(n-3)/2}\, \delta(q_{\alpha\beta} - m^2 xy)$$
$$= \frac{\Gamma(n/2)^2}{\pi\Gamma((n-1)/2)^2} \int_{-1}^1 dx\, (1-x^2)^{(n-3)/2}$$
$$\times \left(1 - \frac{q_{\alpha\beta}^2}{m^4 x^2}\right)^{(n-3)/2} \frac{\theta(x^2 - q_{\alpha\beta}^2/m^4)}{m^2\,|x|}$$
$$= \frac{\theta(1 - q_{\alpha\beta}^2/m^4)\,\Gamma(n/2)^2}{\pi m^2 \Gamma((n-1)/2)^2} \int_{q_{\alpha\beta}^2/m^4}^1 dt\, (1-t)^{(n-3)/2}$$
$$\times \left(t - \frac{q_{\alpha\beta}^2}{m^4}\right)^{(n-3)/2} t^{-(n-1)/2} \tag{5.5}$$

where $\theta(x)$ is the Heaviside step function, $\theta(x) = 1$ for $x > 0$ and $\theta(x) = 0$ for $x < 0$. By using Eq. (2.2.6.1) from ref. 6, this becomes

$$P_{\alpha\beta}(q_{\alpha\beta}) = \theta\left(1 - \frac{q_{\alpha\beta}^2}{m^4}\right) \frac{\Gamma(n/2)^2}{\pi m^2 \Gamma(n-1)}$$
$$\times \left(1 - \frac{q_{\alpha\beta}^2}{m^4}\right)^{n-2} \left(\frac{q_{\alpha\beta}^2}{m^4}\right)^{-(n-1)/2}$$
$$\times\ {}_2F_1\left(\frac{n-1}{2}, \frac{n-1}{2}; n-1; 1 - \frac{m^4}{q_{\alpha\beta}^2}\right) \qquad (n = 2, 3, \ldots) \tag{5.6}$$

Because of the special form of the hypergeometric function, this result can also be written in terms of associated Legendre functions of the second kind Q_ν,

$$P_{\alpha\beta}(q_{\alpha\beta}) = \theta\left(1 - \frac{q_{\alpha\beta}^2}{m^4}\right) \frac{2^{n-1}\Gamma(n/2)^3}{\pi^{3/2} m^2 \Gamma(n-1)\,\Gamma((n-1)/2)} \left(1 - \frac{q_{\alpha\beta}^2}{m^4}\right)^{(n-3)/2}$$
$$\times\ Q_{(n-3)/2}\left(\frac{1 + q_{\alpha\beta}^2/m^4}{1 - q_{\alpha\beta}^2/m^4}\right) \qquad (n = 2, 3, \ldots) \tag{5.7}$$

where we have used Eq. (7.3.1.71) from ref. 6 and Eq. (8.2.4) from ref. 5.

Equation (5.7) means that below T_c and for odd n, $P_{\alpha\beta}$ can be expressed as

$$\theta(1 - q_{\alpha\beta}^2/m^4)[R_1(q_{\alpha\beta}) \ln q_{\alpha\beta} + R_2(q_{\alpha\beta})]$$

(R_1, R_2 rational functions), while for even n it can be expressed in terms of complete elliptic integrals. In both cases, $P_{\alpha\beta}$ has a logarithmic singularity at $q_{\alpha\beta} = 0$.

The diagrams in Fig. 1 illustrate the behavior of $P_{\alpha\beta}(q_{\alpha\beta})$ above and below T_c.

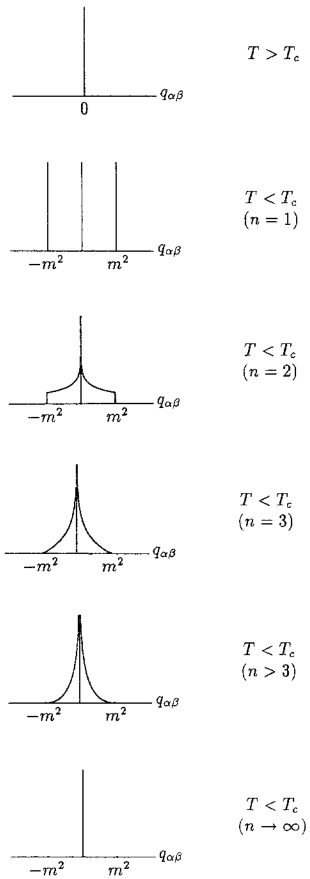

Fig. 1. Probability distribution $P_{\alpha\beta}(q_{\alpha\beta})$ for the overlap between pure states. $P_{\alpha\beta}(q_{\alpha\beta})$ becomes a delta function as $n \to \infty$.

6. PROBABILITY DISTRIBUTION $W_{\alpha\beta}$ FOR THE SELF-CORRELATION $d_{\alpha\beta}$

From Eq. (4.14) we get the self-correlation $d_{\alpha\beta}(r)$ defined in (3.1),

$$d_{\alpha\beta}(r) = \begin{cases} \delta_{\alpha\beta} & (T > T_c) \\ \dfrac{1}{2J}\delta_{\alpha\beta} + n\hat{B}_r^\alpha \hat{B}_r^\beta \left(1 - \dfrac{1}{2J}\right) & (T < T_c) \end{cases} \tag{6.1}$$

The probability distribution $W_{\alpha\beta}$ of $d_{\alpha\beta}$ is defined in (3.2). If we take the remarks at the beginning of Section 5 into account, $W_{\alpha\beta}$ can be derived in a similar fashion as $P_{\alpha\beta}$.

6.1. $W_{\alpha\beta}(d_{\alpha\beta})$ for $T > T_c$

By inserting (6.1) and (5.2) into the definition (3.2) for $W_{\alpha\beta}$ we get

$$W_{\alpha\beta}(d_{\alpha\beta}) = \sum_r P_r \delta(d_{\alpha\beta} - \delta_{\alpha\beta}) = \delta(d_{\alpha\beta} - \delta_{\alpha\beta}) \tag{6.2}$$

6.2. $W_{\alpha\beta}(d_{\alpha\beta})$ for $T < T_c$

By inserting (6.1) and (5.2) into the definition (3.2) for $W_{\alpha\beta}$, and by using n-dimensional spherical coordinates, we have to distinguish the cases $\alpha = \beta$ and $\alpha \neq \beta$.

6.2.1. $\alpha = \beta$. For this case

$$W_{\alpha\alpha}(d_{\alpha\alpha}) = \frac{\Gamma(n/2)}{2\pi^{n/2}} \int_0^{2\pi} d\theta_1 \prod_{k=2}^{n-1} \int_0^{\pi} d\theta_k \sin^{k-1}\theta_k \times \delta\left[d_{\alpha\alpha} - \frac{1}{2J} - n\left(1 - \frac{1}{2J}\right)\cos^2\theta_{n-1}\right] \tag{6.3}$$

where, without loss of generality, we have chosen the x_n axis of the spherical coordinate system in the direction α. Thus

$$\begin{aligned} W_{\alpha\alpha}(d_{\alpha\alpha}) &= \frac{\Gamma(n/2)}{\pi^{1/2}((n-1)/2)} \int_{-1}^{1} dx\, (1 - x^2)^{(n-3)/2} \times \delta\left[d_{\alpha\alpha} - \frac{1}{2J} - n\left(1 - \frac{1}{2J}\right)x^2\right] \\ &= \frac{\theta(d_{\alpha\alpha} - 1/(2J))\,\Gamma(n/2)}{\pi^{1/2}\Gamma((n-1)/2)} \int_{-1}^{1} dx\, (1 - x^2)^{(n-3)/2} \times \left[\frac{\delta(x - z_1)}{z_2} + \frac{\delta(x + z_1)}{z_2}\right] \end{aligned} \tag{6.4}$$

with

$$z_1 \equiv \left[\frac{d_{\alpha\alpha} - 1/(2J)}{n(1 - 1/(2J))}\right]^{1/2}, \qquad z_2 \equiv \left[4n\left(1 - \frac{1}{2J}\right)\left(d_{\alpha\alpha} - \frac{1}{2J}\right)\right]^{1/2} \tag{6.5}$$

Here we have used the fact that $(1 - 1/(2J)) > 0$ for $T < T_c$. Equation (6.4) becomes

$$W_{\alpha\alpha}(d_{\alpha\alpha}) = \frac{2\Gamma(n/2)}{\pi^{1/2}\Gamma((n-1)/2)}\,\theta\left(d_{\alpha\alpha} - \frac{1}{2J}\right) \times \theta(1 - z_1^2)\,\frac{(1 - z_1^2)^{(n-3)/2}}{z_2} \qquad (n = 2, 3,...) \tag{6.6}$$

6.2.2. α ≠ β. Without loss of generality, we can choose the axis $\hat{x}_n$ of the spherical coordinate system along the direction α, and the basis vector $\hat{x}_{n-1}$ along the direction β. With this choice, and for $n > 2$, we have

$$\begin{aligned} W_{\alpha\alpha}(d_{\alpha\alpha}) &= \frac{\Gamma(n/2)}{2\pi^{n/2}} \int_0^{2\pi} d\theta_1 \prod_{k=2}^{n-1} \int_0^{\pi} d\theta_k \sin^{k-1}\theta_k \\ &\quad \times \delta\left[d_{\alpha\beta} - n\left(1 - \frac{1}{2J}\right)\cos\theta_{n-2}\cos\theta_{n-1}\sin\theta_{n-1}\right] \\ &= \frac{n-2}{2\pi}\int_{-1}^{1} dx \int_{-1}^{1} dy\,(1 - x^2)^{(n-4)/2}(1 - y^2)^{(n-3)/2} \\ &\quad \times \delta\left[d_{\alpha\beta} - n\left(1 - \frac{1}{2J}\right)xy(1 - y^2)^{1/2}\right] \\ &= \frac{n-2}{2\pi n(1 - 1/(2J))}\int_{-1}^{1}\frac{dy}{|y|}\left[1 - y^2 - \frac{d_{\alpha\beta}^2}{n^2(1 - 1/(2J))^2 y^2}\right]^{(n-4)/2} \\ &\quad \times \theta\left[1 - \frac{d_{\alpha\beta}^2}{n^2(1 - 1/(2J))^2 y^2(1 - y^2)}\right] \\ &= \frac{(n-2)\,\theta(z)}{2\pi n(1 - 1/(2J))}\int_{(1-\sqrt{z})/2}^{(1+\sqrt{z})/2} dx\, x^{-(n-2)/2}\left(x - \frac{1 - \sqrt{z}}{2}\right)^{(n-4)/2} \\ &\quad \times \left(\frac{1 + \sqrt{z}}{2} - x\right)^{(n-4)/2} \end{aligned} \tag{6.7}$$

where we have defined the quantity

$$z \equiv 1 - \frac{4d_{\alpha\beta}^2}{n^2(1 - 1/(2J))^2} \tag{6.8}$$

Equation (6.7) can be evaluated by using the relation (2.2.6.1) from ref. 6. We find

$$W_{\alpha\beta}(d_{\alpha\beta}) = \frac{(n-2)\,\Gamma((n-2)/2)^2}{2\pi n(1-1/(2J))\,\Gamma(n-2)}\,\theta(z)\,z^{(n-3)/2}\left(\frac{1-\sqrt{z}}{2}\right)^{-(n-2)/2}$$
$$\times\,{}_2F_1\left(\frac{n-2}{2},\frac{n-2}{2};n-2;\frac{2\sqrt{z}}{\sqrt{z}-1}\right)\qquad(\alpha\neq\beta;\,n=3,4,\ldots)\tag{6.9}$$

This result can also be written in terms of associated Legendre functions of the second kind Q_ν by using Eq. (7.3.1.71) from ref. 6 and Eq. (8.2.4) from ref. 5,

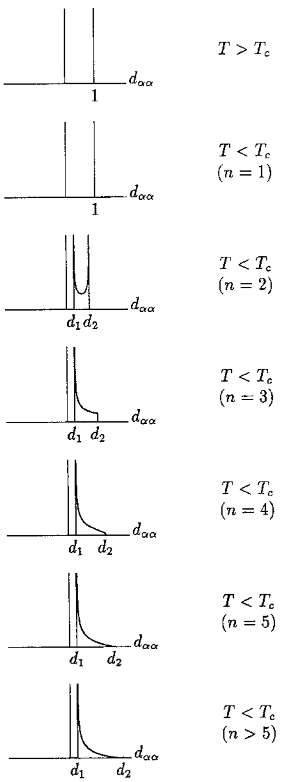

Fig. 2. Probability distribution $W_{\alpha\beta}(d_{\alpha\beta})$ for $\alpha=\beta$. The minimum and maximum self-correlations $d_{\alpha\alpha}$ for the spin components are $d_1 = 1/(2J)$ and $d_2 = n-(n-1)/(2J)$, respectively. Note that $W_{\alpha\alpha}(d_{\alpha\alpha})$ does not become a delta function as $n\to\infty$. The graph for $n>5$ is understood as showing the qualitative behavior of $W_{\alpha\alpha}$ when $n>5$.

$$W_{\alpha\beta}(d_{\alpha\beta}) = \frac{(n-2)\, 2^{n-3} \Gamma((n-2)/2)\, \Gamma((n-1)/2)}{n\pi^{3/2}(1-1/(2J))\, \Gamma(n-2)}\, \theta(z)\, z^{(n-4)/4} \times Q_{(n-4)/2}\left(\frac{1}{\sqrt{z}}\right) \qquad (\alpha \neq \beta;\, n = 3, 4,\ldots) \tag{6.10}$$

For $n = 2$, on the other hand, we have

$$W_{\alpha\beta}(d_{\alpha\beta}) = \frac{1}{2\pi} \int_0^{2\pi} d\phi\, \delta\left[d_{\alpha\beta} - 2\left(1 - \frac{1}{2J}\right) \cos\phi \sin\phi \right] = \frac{\theta(z)}{\pi(1-1/(2J))\sqrt{z}} \qquad (\alpha \neq \beta;\, n = 2) \tag{6.11}$$

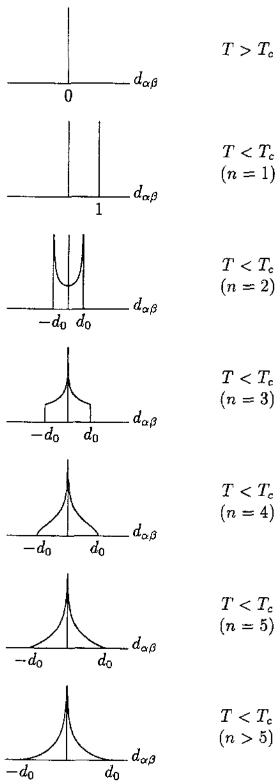

Fig. 3. Probability distribution $W_{\alpha\beta}(d_{\alpha\beta})$ for $\alpha \neq \beta$. The maximum self-correlation between different spin components of the same spin is $d_0 = (n/2)[1 - 1/(2J)]$. Note that $W_{\alpha\beta}(d_{\alpha\beta})$ does not become a delta function as $n \to \infty$. Again, the graph for $n > 5$ is understood as illustrating the qualitative behavior of $W_{\alpha\beta}$ when $n > 5$.

where z was defined in (6.8). For $n=1$, we have trivially

$$W_{\alpha\beta}(d_{\alpha\beta})=\delta(d_{\alpha\beta}-1) \qquad (n=1) \tag{6.12}$$

Equations (6.10) and (6.11) mean that below T_c and for odd $n>1$, $W_{\alpha\beta}$ can be expressed in terms of complete elliptic integrals, while for even n, $W_{\alpha\beta}$ is of the form

$$\theta(\sqrt{z})\left[R_1(\sqrt{z})\ln\left(\frac{\sqrt{z}+1}{\sqrt{z}-1}\right)+R_2(\sqrt{z})\right]$$

R_1 and R_2 are rational functions with $R_1=0$ for $n=2$ and $R_1\neq 0$ for $n>2$. Thus, for $n>2$, $W_{\alpha\beta}$ has a logarithmic singularity at $d_{\alpha\beta}=0$, while for $n=2$ it has poles at $d_{\alpha\beta}=\pm[1-1/(2J)]^{1/2}$.

The diagrams in Figs. 2 and 3 illustrate the behavior of $W_{\alpha\beta}(d_{\alpha\beta})$ above and below T_c.

7. PROBABILITY DISTRIBUTION P FOR THE OVERLAP q

From Eq. (4.1) we find that the overlap $q(r,t)$ defined in (3.1) is given by

$$q(r,t)=\begin{cases}0 & (T>T_c)\\ m^2\hat{\mathbf{B}}_r^T\hat{\mathbf{B}}_t & (T<T_c)\end{cases} \tag{7.1}$$

The probability distribution $P(q)$ is defined in Eq. (3.2). If we take the remarks at the beginning of Section 5 into account, $P(q)$ can be derived in the same fashion as $P_{\alpha\beta}(q_{\alpha\beta})$.

7.1. $P(q)$ for $T>T_c$

By inserting (7.1) and (5.2) into the definition (3.2) for $P(q)$, we get

$$P(q)=\sum_{r,t}P_rP_t\delta(q-0)=\delta(q) \tag{7.2}$$

7.2. $P(q)$ for $T<T_c$

By inserting (7.1) and (5.2) into the definition (3.2) for $P(q)$ and by using n-dimensional spherical coordinates, we have

$$\begin{aligned}P(q) &= \sum_{r,t} P_r P_t \delta(q - m^2 \hat{\mathbf{B}}_r^T \hat{\mathbf{B}}_t) \\ &= \left[\frac{\Gamma(n/2)}{2\pi^{n/2}}\right]^2 \int_0^{2\pi} d\theta_1 \int_0^{2\pi} d\psi_1 \prod_{k=2}^{n-1} \int_0^{\pi} d\theta_k \sin^{k-1}\theta_k \\ &\quad \times \int_0^{\pi} d\psi_k \sin^{k-1}\psi_k \delta(q - m^2 \cos\psi_{n-1}) \\ &= \frac{\Gamma(n/2)}{\pi^{1/2}\Gamma((n-1)/2)} \int_{-1}^{1} dx\,(1-x^2)^{(n-3)/2}\,\delta(q - m^2 x) \qquad (7.3)\end{aligned}$$

where, without loss of generality, we have associated the θ coordinate system with $\hat{\mathbf{B}}_r$ and the ψ coordinate system with $\hat{\mathbf{B}}_t$, and we have chosen the x_n axis of the ψ coordinate system in the direction $\hat{\mathbf{B}}_r$. Thus,

$$P(q) = \frac{\Gamma(n/2)}{\pi^{1/2} m^2 \Gamma((n-1)/2)}\,\theta\left(1 - \frac{q^2}{m^4}\right)\left(1 - \frac{q^2}{m^4}\right)^{(n-3)/2} \qquad (n = 2, 3, \ldots) \qquad (7.4)$$

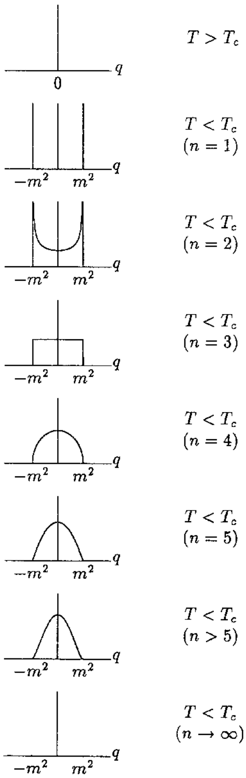

Fig. 4. Probability distribution $P(q)$ for the overlap between pure states. $P(q)$ becomes a delta function as $n \to \infty$.

The diagrams in Fig. 4 illustrate the behavior of $P(q)$ above and below T_c. Note that for $n=2$, $P(q)$ peaks at the extremal values $q = \pm m^2$, while the related distribution $P_{\alpha\beta}(q_{\alpha\beta})$ for $n=2$ peaks at $q=0$. Further, $P(q)$ follows the semicircle law for $n=4$.

ACKNOWLEDGMENTS

T.T. is very grateful for the financial support of the Australian Postgraduate Research Award Scheme.

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